How many family-independent, independent U(1)'s, can we gauge?

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In the context of $SU(3)_c\otimes SU(2)_L\otimes [U(1)]^n$, $SU(3)_c\otimes SU(2)_L\otimes SU(2)_R\otimes [U(1)]^{n'}$ and $SU(3)_c\otimes SU(3)_L\otimes [U(1)]^{n''}$, we analyze the possible values for n, n' and n''. That is, we look for the number of family-independent, independent abelian hypercharges that can be gauged in each one of those models. We find that $n = n' = 1$ for the minimal fermion content, but n can take any value if we add the right-handed neutrino field to each family in the Standard Model.

I. INTRODUCTION

The standard model (SM) local gauge group $SU(3)_c\otimes SU(2)_L\otimes U(1)_Y\equiv G_{SM}$ with $SU(2)_L\otimes U(1)_Y$ hidden and $SU(3)_c$ confined, has been very sucessfull in explaining a huge amount of experimental data related with particle interactions at low energies. Nevertheless, phycisists have been looking for variants of this model in order to explain several experimental facts such as parity violation, the mass spectrum of the known elementary fermions, or the explanation of different anomalies, in the context of extended electroweak theories as for example $SU(2)_L\otimes SU(2)_R\otimes U(1)_{(B-L)}$ or $SU(2)_L\otimes U(1)_X\otimes U(1)_Z$, etc..

More profound questions such as why charge is quantized [1] or why the low energy gauge group is the one belonging to the SM [2] are still unanswered. In this note we are going to refer to a different fundamental question which is related to the number of family-independent, independent local gauge Abelian groups that can be gauged simultaneously.

II. STANDARD MODEL

First let us start our analysis in the context of a family-independent $U(1)$ Abelian factor and look for the constraints that anomaly cancellation imposes on such a model in an environment provided by the SM gauge group $SU(3)_c\otimes SU(2)_L\otimes U(1)_Y$. The anomaly cancellation constraint equations are [1]

$$[SU(2)_L]^2U(1): 3Y_q+Y_f=0 \tag{1}$$
$$[SU(3)_c]^2U(1): 2Y_q+Y_u+Y_d=0 \tag{2}$$
$$[grav]^2U(1): 6Y_q+3Y_u+3Y_d+2Y_f+Y_e=0 \tag{3}$$
$$[U(1)]^3: 6Y_q^3+3Y_u^3+3Y_d^3+2Y_f^3+Y_e^3=0 \tag{4}$$

where $Y_\eta(\eta=q,f,u,d,e)$ are the $U(1)_Y$ hypercharges of the following multiplets: $\psi_q^T=(u,d)_L\sim[3,2]$, $\psi_f^T=(\nu,e^-)_L\sim[1,2]$, $\psi_u=u_L^c\sim[\bar{3},1]$, $\psi_d=d_L^c\sim[\bar{3},1]$ and $\psi_e=e_L^+\sim[1,1]$ which are the fifteen states belonging to one family (the upper c symbol stands for charge conjugation and the numbers in the square bracket stand for the $[SU(3)_c, SU(2)_L]$ quantum numbers).

By combining the cubic equation with the other three lineal ones we get:

$$Y_q(2Y_q-Y_u)(4Y_q+Y_u)=0, \tag{5}$$

which provides three independent families of solutions as quoted in Table I, where the SM hypercharge value has been included for comparison.

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Sol.	Y_q	Y_f	Y_e	Y_u	Y_d
A	0	0	0	a	−a
B	b	−3b	6b	2b	−4b
C	c	−3c	6c	−4c	2c
SM	1/3	−1	2	−4/3	2/3

In Table I a, b and c are real arbitrary parameters, and the SM hypercharge is related with solution **C**, for $c = 1/3$.

From Table I we realize that charge is not quantized in the context of the SM [1]. Not only there are three different solutions to the anomaly constraint equations, but we do not know why solution **C** is the one chosen by nature, and even worse, we can not explain why $c = 1/3$ for that particular solution [1].

Now, taking into account all three solutions, namely $U(1)_{Y_A}$, $U(1)_{Y_B}$ and $U(1)_{Y_C}$ one can ask whether $U(1)_{Y_A} \otimes U(1)_{Y_B} \otimes U(1)_{Y_C}$ can be gauged simultaneous, or at least two of them at the same time. In order to answer the second question we must look for solution to the following two new anomaly constraint equations:

$$6Y_{q\alpha}^2 Y_{q\beta} + 3Y_{u\alpha}^2 Y_{u\beta} + 3Y_{d\alpha}^2 Y_{d\beta} + 2Y_{f\alpha}^2 Y_{f\beta} + Y_{e\alpha}^2 Y_{e\beta} = 0 \tag{6}$$

$$6Y_{q\alpha} Y_{q\beta}^2 + 3Y_{u\alpha} Y_{u\beta}^2 + 3Y_{d\alpha} Y_{d\beta}^2 + 2Y_{f\alpha} Y_{f\beta}^2 + Y_{e\alpha} Y_{e\beta}^2 = 0 \tag{7}$$

for $\alpha, \beta = \mathbf{A}, \mathbf{B}, \mathbf{C}$. The only solution to Eqs.(6) and (7) is for $\alpha = \beta$, with a, b and c still free parameters. So, only one solution to the anomaly constraint equations can be gauged, and once it has been gaugen it can be gauged as many times as we want; but we can not gauge two different solutions at the same time [3].

From the previous analysis we conclude that the hypercharges Y_A and Y_B should be automatically excluded once Y_C, with $c = 1/3$, is gauged as the correct hypercharge chosen by nature. So, in the context of $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ there is only one anomaly free independent hypercharge corresponding to $Y = Y_C$ with $c = 1/3$. That hypercharge corresponds to the SM hypercharge, and it can be gauged as many times as we wishes allowing the value for c to be a free parameter.

III. SM PLUS RIGHT-HANDED NEUTRINO

When the right-handed neutrino $\psi_\nu = \nu_L^c \sim [1, 1]$ is included in each family as part of the spectrum, the set of anomaly constraint equations become:

$$[SU(2)_L]^2 U(1) : 3Y_q + Y_f = 0 \tag{8}$$
$$[SU(3)_c]^2 U(1) : 2Y_q + Y_u + Y_d = 0 \tag{9}$$
$$[grav]^2 U(1) : 6Y_q + 3Y_u + 3Y_d + 2Y_f + Y_e + Y_\nu = 0 \tag{10}$$
$$[U(1)]^3 : 6Y_q^3 + 3Y_u^3 + 3Y_d^3 + 2Y_f^3 + Y_e^3 + Y_\nu^3 = 0 \tag{11}$$

where Y_ν is the $U(1)_Y$ hypercharge associated with the right-handed neutrino field. Combining again the cubic equation with the three lineal ones we get:

$$Y_q(2Y_q - Y_u - Y_\nu)(4Y_q + Y_u - Y_\nu) = 0, \tag{12}$$

which again provide a set of three independent families of solutions as presented in Table II (notice by the way that Eq. (12) reduces to Eq. (5) for $Y_\nu = 0$ as it should).

Sol.	Y_q	Y_f	Y_e	Y_ν	Y_u	Y_d
A'	0	0	η	$-\eta$	δ	$-\delta$
B'	(a+b)/2	−3(a+b)/2	3a+2b	b	a	−(2a+b)
C'	−(A−B)/4	3(A−B)/4	(B−3A)/2	B	A	−(A+B)/2

where η, δ, a, b, A and B are real arbitrary parameters. Notice that solution **B'** is independent of solution **A'** for $b \neq -a$ and solution **C'** is independent of solution **A'** for $B \neq A$. From Table II we can see that the SM hypercharge value is related with solution **C'** for $B = 0$ and $A = -4/3$, and the $B - L$ (Baryon number minus Lepton number) hypercharge is related with solution **B'** for $a = -1/3$ and $b = 1$. Again, charge is not quantized in this simple extension of the SM.

But can we now gauge $U(1)_{Y_{A'}} \otimes U(1)_{Y_{B'}} \otimes U(1)_{Y_{C'}}$ simultaneously? Or at least two of them at the same time? To answer the second question we must look for solutions to the following two new anomaly constraint equations

$$6Y_{q\alpha}^2 Y_{q\beta} + 3Y_{u\alpha}^2 Y_{u\beta} + 3Y_{d\alpha}^2 Y_{d\beta} + 2Y_{f\alpha}^2 Y_{f\beta} + Y_{e\alpha}^2 Y_{e\beta} + Y_{\nu\alpha}^2 Y_{\nu\beta} = 0 \tag{13}$$

$$6Y_{q\alpha} Y_{q\beta}^2 + 3Y_{u\alpha} Y_{u\beta}^2 + 3Y_{d\alpha} Y_{d\beta}^2 + 2Y_{f\alpha} Y_{f\beta}^2 + Y_{e\alpha} Y_{e\beta}^2 + Y_{\nu\alpha} Y_{\nu\beta}^2 = 0, \tag{14}$$

for $\alpha, \beta =$ **A',B'** and **C'**. Solutions to this set of equations for the three possible combinations of values for α and β are presented in Table III.

	A'	**B'**	**C'**
A'	Trivial	$b = -a$; η, δ arbitrary (Trivial) $\eta = \delta$, a, b arbitrary	$B = A$; η, δ arbitrary (Trivial) $\eta = -\delta$, A, B arbitrary
B'		Trivial	$B = -3A$, a, b arbitrary $b = -3a$, A, B arbitrary
C'			Trivial

From Table III we see that we can gauge one of the three solutions at least two times (what we call the Trivial solutions in the Table), but what is more important now is that, contrary to the SM situation, we can gauge any two different solutions at the same time.

But, can we gauge the three solutions at the same time? To answer this question the further anomaly constraint equation must also be satisfied

$$\begin{aligned}&6Y_{qA'}Y_{qB'}Y_{qC'} + 3Y_{uA'}Y_{uB'}Y_{uC'} + 3Y_{dA'}Y_{dB'}Y_{C'} \\ &\quad +2Y_{fA'}Y_{fB'}Y_{C'} + Y_{eA'}Y_{eB'}Y_{C'} + Y_{\nu A'}Y_{\nu B'}Y_{\nu C'} = 0.\end{aligned} \tag{15}$$

The algebra shows now the following solutions: $A' = B' = C'$ (Trivial), or $A' = B'$, or $A' = C'$ or $B' = C'$. So, we can gauge one of the three familes of solutions three or more times (in fact as many times as we wishes), which in turn provides an infinity number of independent $U(1)'s$ that can be gauged simultaneously; also we can gauge two independent families at the same time, but we can not gauge the three independent families simultaneously.

Our conclusion is now that once we gauge solution **C'** for $B = 0$ and $A = -4/3$ (the SM hypercharge value) we can gauge again solution **C'** as many times as we want with A and B free parameters, and still we can gauge either solutions **A'** for $\eta = -\delta$ or either solution **B'** for $b = -3a$. In particular we can gauge the hypercharge $B - L$ simultaneously with the SM hypercharge Y. This new amazing result is a consequence of introducing the right-handed neutrino field and it is a novelty not present in the SM.

IV. THE LEFT-RIGHT SYMMETRIC MODEL

The Left-Right symmetric model of the electroweak interactions was introduced in the middle of the seventies [4] in order to elucidate the origin of parity violation in low energy physics. Within the framework of local gauge theories the idea was to enlarge the SM local gauge group G_{SM} to the Left-Right symmetric one $SU(3)_c \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_X \equiv G_{LR}$, where in the literature $X = B - L$ [4].

The right-handed neutrino is automatically included as an isopartner of e_R in G_{LR}. The anomaly constraint equations on X are now:

$$\begin{aligned}[SU(2)_L]^2U(1) &: 3X_{qL} + X_{fL} = 0 \\ [SU(2)_R]^2U(1) &: 3X_{qR} + X_{fR} = 0 \\ [SU(3)_c]^2U(1) &: X_{qL} - X_{qR} = 0 \\ [grav]^2U(1) &: 6X_{qL} - 6X_{qR} + 2X_{fL} - 2X_{fR} = 0 \\ [U(1)]^3 &: 6X_{qL}^3 - 6X_{qR}^3 + 2X_{fL}^3 - 2X_{fR}^3 = 0\end{aligned}$$

where $X_\eta(\eta = qL, qR, fL, fR)$ are the $U(1)_X$ hypercharges of the following multiplets: $\psi^T_{qL} = (u,d)_L,\ \psi^T_{qR} = (u,d)_R,\ \psi^T_{fL} = (\nu,e)_L,\ \psi^T_{fR} = (\nu,e)_R$.

The above equations imply the unique solution $X_{qR} = X_{qL}$ and $X_{fL} = X_{fR} = -3X_{qL}$ (i.e. the $U(1)_X$ is vectorlike). So, there is only one independent family of hypercharge solutions in G_{LR} (for $X_{qL} = 1/3$ it is the $B-L$ hypercharge). Again, there is not charge quantization in G_{LR} because X_{qL} is a free parameter.

Now let us analyze the group $G_{LR} \otimes U(1)_Z$. The new anomaly constraint equations to be considered are:

$$\begin{aligned}
[U(1)_X]^2U(1)_Z &: 6X^2_{qL}Z_{qL} - 6X^2_{qR}Z_{qR} + 2X^2_{fL}Z_{fL} - 2X^2_{fR}Z_{fR} = 0 \\
[U(1)_Z]^2U(1)_X &: 6Z^2_{qL}X_{qL} - 6Z^2_{qR}X_{qR} + 2Z^2_{fL}X_{fL} - 2Z^2_{fR}X_{fR} = 0
\end{aligned}$$

The solution to the former set of equations (with $Z_{qR} = Z_{qL}$ and $Z_{fL} = Z_{fR} = -3Z_{qL}$) is only the trivial one $X_\eta = Z_\eta$ for $\eta = qL, qR, fL, fR$. Again, we can gauge two or more $U(1)'s$ in the context of $SU(2)_L \otimes SU(2)_R \otimes SU(3)_c$, as far the hypercharge of those $U(1)'s$ are all of them proportional to $B-L$ and, as in the SM case, they are not independent.

V. THE (3,3,1) MODEL

Finally let us consider the gauge group $SU(3)_L \otimes SU(3)_c \otimes U(1)_K \equiv G_{331}$ as an extension of the SM group [5], and let us study the anomaly constraint equations for one single family of quarks and leptons as it is done for the SM.

For one family let us start with $\psi^T_L = (u,d,q)_L$ with G_{331} quantum numbers $(3,3,K_Q)$, where q_L is an exotic quark of electric charge to be fixed ahead. Now, in order to have $SU(3)_c$ vectorlike as in the SM we must introduce the fields $\psi_u = u^c_L,\ \psi_d = d^c_L$ and $\psi_q = q^c_L$ with G_{331} quantum numbers given by $(1,\bar{3},K_u), (1,\bar{3},K_d)$ and $(1,\bar{3},K_q)$ respectively. Next, in order to introduce the lepton fields and to cancel the $SU(3)_L$ anomaly we must define three new multiplets $\chi^T_{1L} = (e^-,\nu,l_1)_L,\ \chi^T_{2L} = (l_2,l_3,l_4)_L$ and $\chi^T_{3L} = (l_5,l_6,l_7)_L$ with G_{331} quantum numbers $(\bar{3},1,K_1)$, $(\bar{3},1,K_2)$ and $(\bar{3},1,K_3)$ respectively, where $l_i,\ i = 1,...7$ are related to lepton fields, most of them exotics.

The anomaly constraint equations on K are now:

$$\begin{aligned}
[SU(3)_c]^2U(1) &: 3K_Q + K_u + K_d + K_q = 0 \\
[SU(3)_L]^2U(1) &: 3K_Q + K_1 + K_2 + K_3 = 0 \\
[grav]^2U(1) &: 9K_Q + 3K_u + 3K_d + 3K_q + 3K_1 + 3K_2 + 3K_3 = 0 \\
[U(1)]^3 &: 9K^3_Q + 3K^3_u + 3K^3_d + 3K^3_q + 3K^3_1 + 3K^3_2 + 3K^3_3 = 0.
\end{aligned}$$

The former set of equations imply $K_Q = 0$ and $K_uK_dK_q = -K_1K_2K_3$ which in turn imply an infinity set of independent solutions. In order to reduce the number of independent solutions and to construct a reasonable model, further constraints must be introduced, so let us look for solutions to the former set of equations such that $K_q = K_d$ and $K_1 = K_2$ (notice that the anomaly constraint equations on K are $K_u \leftrightarrow K_d$ and $K_1 \leftrightarrow K_2$ symmetric); with these conditions we warrant that not exotic electric charges are present on the model; using them we get $K_q = K_d \equiv K = -K_u/2 = -K_1 = -K_2 = K_3/2$. Again, K is a free parameter and charge is not quantized.

Using for the symmetry breaking chain $SU(3)_L \longrightarrow SU(2)_L \otimes U(1)_S$ the branching rule $3 \longrightarrow 2(s) + 1(-2s)$, and the electric charge generator defined by $Q = T_3 + S + K$ where $T_3 = \pm 1/2$ is the third isospin component, we have for $s = 1/6$ the following electric charges: $Q_u = 2/3,\ Q_d = Q_q = -1/3$ and $\chi^T_{1L} = (e^-,\nu,N^0_1)_L,\ \chi^T_{2L} = (E^-,N^0_2,N^0_3)_L$ and $\chi^T_{3L} = (N^0_4,E^+,e^+)_L$, where $N^0_i,\ i = 1,2,3,4$ are four exotic neutrino fields and E is an exotic electron. The 27 fields in $\psi_L, \psi_u, \psi_d, \psi_q$ and $\chi_{iL},\ i = 1,2,3$ are nothing else but the 27 fields in the fundamental representation of the grand unification group E_6 [6]. As a matter of fact, the group representation that we have derived in this exercise is such that $G_{331} \subset E_6$ [7], but of course there are many possible realistic models in the context of the gauge group G_{331} [5].

VI. FLIPPED $SU(5) \otimes U(1)$

In this section we analyzed the flipped $SU(5) \otimes U(1)_W$ model [8] as an application of the formalism presented in Section 3. This model unifies the 16 fields in one family at a high scale, and yet has an extra $U(1)$ quantum number which does not coincide with the SM hypercharge Y. In this model

the minimal choice of multiplets that are free of the $[SU(5)]^3$ anomaly are [8] $(10,\ W_{10})$, $(\bar{5},\ W_5)$ and $(1,\ W_1)$. The other anomaly constraint equations are:

$$\begin{aligned} [SU(5)]^2U(1) &: W_5 + 3W_{10} = 0 \\ [grav]^2U(1) &: 5W_5 + 10W_{10} + W_1 = 0 \\ [U(1)]^3 &: 5W_5^3 + 10W_{10}^3 + W_1^3 = 0, \end{aligned} \tag{16}$$

which imply $W_{10} = -W_5/3 = W_1/5$ with W_{10} a free parameter (again charge is not quantized in this model either).

A simple check shows that this $U(1)_W$ is just a particular case of solution **C'** in the Table II in Section 3, with $A = W_{10}$ and $B = 5W_{10}$. (This same $U(1)_W$ is the one present in the grand unification group $SO(10)$ [9] for the breaking chain $SO(10) \longrightarrow SU(5) \otimes U(1)_W$ which has the branching rule $16 \longrightarrow 1(-5) + \bar{5}(3) + 10(-1)$.)

VII. ACKNOWLEDGMENTS

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